Compactifications

Recall from the homework: $\mathbb{R}^n$ is not compact,

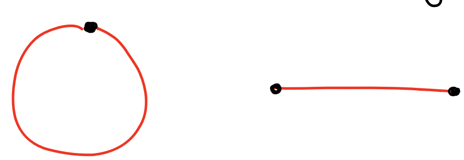
but $\mathbb{R}^n \cup \{\infty\}$ w/ basis given by open balls and
$U_r = \{x \in \mathbb{R}^n \mid |x| > r\} \cup \{\infty\}$ for $r > 0$ is compact.
This is called a compactification of $\mathbb{R}^n$.

More generally...

Def: If Y is compact (Hausdorff) + $X \hookrightarrow Y$ is a homeomorphism onto its image such that X is dense in Y, Y is a compactification of X. If $Y \setminus X$ is a single point, Y is a one-point compactification of X.

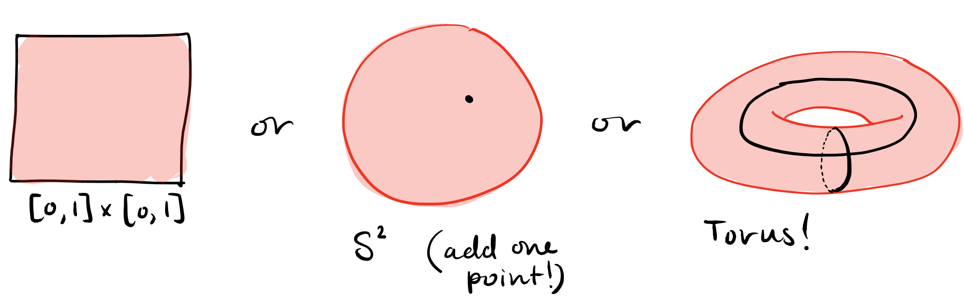
Examples

1.) S^1 is a compactification of $(0,1)$, but so is $[0,1]$ — i.e. compactifications are not necessarily unique.



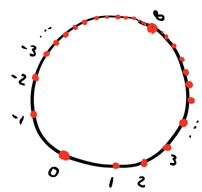
2.) $(0,1) \times (0,1)$ has lots of compactifications:

e.g.



3.) $\mathbb{RP}^n$ (resp. $\mathbb{CP}^n$) (projective n-space) is a compactification of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$).

4.) Let $\mathbb{Z}$ be given the discrete topology. Let $X = \mathbb{Z} \cup \{\infty\}$, given the subspace topology in $\mathbb{R} \cup \{\infty\}$. This gives a one-point compactification of $\mathbb{Z}$.



Compactifications are really useful! For example, in algebraic geometry, compact varieties are much easier to work with.

When do compactifications exist?

Local compactness

Def: X is locally compact at x if there is some compact $C \subseteq X$ that contains a neighborhood of X.

X is locally compact if its locally compact at every point

Examples

1.) Any compact space is locally compact.

2.) $\mathbb{R}^n$ is locally compact: $x \in B_r(x) \subseteq \overline{B_r(x)}$

closed + bounded

3.) $\mathbb{R}^\omega$ is <u>not</u> locally compact since none of its basis elements are contained in compact subspaces (otherwise their closures would be compact).

Local compactness is exactly the necessary and sufficient condition for a Hausdorff space X to have a one-point compactification:

<u>Theorem:</u> X is locally compact Hausdorff $\iff$ $\exists\, Y$ s.t.

1.) X is a subspace of Y

2.) $Y \setminus X$ is a single point.

3.) Y is compact Hausdorff.

<u>Pf:</u> $\Rightarrow$: See Munkres.

$\Leftarrow$: X is a subspace of a Hausdorff space, so it's Hausdorff.

If $x \in X$, Choose $U \ni x$ and $V \supseteq Y \setminus X$ disjoint neighborhoods. Then $X \setminus V$ is closed and thus compact, so $x \in \underset{\substack{\uparrow \\ \text{open}}}{U} \subseteq \underset{\substack{\uparrow \\ \text{compact}}}{X \setminus V} \subset X$.

Thus, X is locally compact. $\square$

<u>Theorem:</u> If Y and Y' are one point compactifications of X, then $\exists$ a homeomorphism between Y and Y' that is the identity on X. i.e.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Y \arrow[rr, "\text{homeo}"] & & Y' \end{tikzcd}$$

commutes

Pf: Let $Y \setminus X = \{p\}$ and $Y' \setminus X = \{q\}$.

Define $f: Y \to Y'$ by $f(x) = \begin{cases} x & \text{if } x \in X \\ q & \text{if } x = p. \end{cases}$

Let $U \subseteq Y$ be open.

If $p \notin U$, $U \subseteq X \Rightarrow f(U) = U$ which is open in X and thus in Y' since $X \subseteq Y'$ is open.

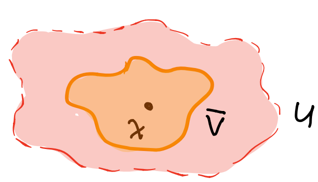
If $p \in U$, then $C = Y \setminus U \subseteq X \subseteq Y$ is closed and thus compact, so $f(C) = C \subseteq X$ is compact and thus closed in Y' since Y' is Hausdorff. Thus $Y' \setminus f(C) = f(U)$ is open, so f is an open map.

By symmetry f^{-1} is open, so f is a homeomorphism. □

The topology on $Y = X \cup \{\infty\}$, the one-pt compactification of X is the collection of open sets in X along w/ sets $U \subseteq Y$ s.t. $\infty \in U$ and $Y \setminus U \subseteq X$ is compact.

If X is Hausdorff, we have another formulation of local compactness that looks more "local":

Thm: X Hausdorff. X is locally compact $\Leftrightarrow$ for $x \in X$ and U a neighborhood of X, $\exists$ a neighborhood V of x s.t. $\overline{V} \subseteq U$ and $\overline{V}$ is compact.

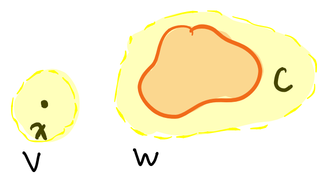


Pf: $\Leftarrow$: Follows from definition.

$\Rightarrow$: Suppose X locally compact. $x \in X$, U a neighborhood of x.

Let Y be the one point compactification of X.

Let $C = Y - U$. Then C is closed in Y and thus compact.

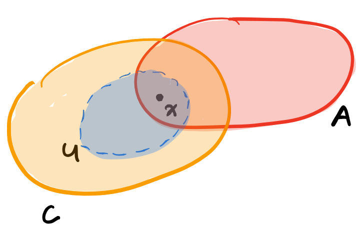


Recall from an earlier proof that we can use the Hausdorff property along w/ compactness of C to find disjoint open sets V and W containing x and C respectively.

Thus $\overline{V} \subseteq U$ is closed and thus compact. $\square$

Cor: X locally compact Hausdorff, $A \subset X$. If A is open or closed it's locally compact.

Pf: If A is closed, then take $x \in A$. Let $C \subseteq X$ be compact containing a neighborhood U of X.



$C \cap A$ is closed in C and thus compact and $U \cap A$ is a neighborhood of

x in A contained in $C \cap A$.

If A is open in X, for $x \in A$, we can choose $x \in V \subseteq A$ s.t. $\overline{V} \subseteq U$ and $\overline{V}$ is compact. $\overline{V}$ contains the neighborhood V of x, so A is locally compact. $\square$